Representation Theory with a Perspective from Category Theory

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1 Introduction

Oftentimes, it is better to understand an algebraic structure by representing its elements as maps on another space. For example, Cayley's Theorem tells us that every finite group is isomorphic to a subgroup of some symmetric group. In particular, representing groups as linear maps on some vector space allows us to translate group theory problems to linear algebra problems. In this paper, we will go over some introductory representation theory, which will allow us to reach an interesting result known as Frobenius Reciprocity. Afterwards, we will examine Frobenius Reciprocity from the perspective of category theory.

2 Representations of Finite Groups

2.1 Basic Definitions

Definition 2.1.1 (Representation)**.** A representation of a group G on a finite-dimensional vector space is a homomorphism $\phi : G \to GL(V)$ where $GL(V)$ is the group of invertible linear endomorphisms on V. The degree of the representation is defined to be the dimension of the underlying vector space. Note that some people refer to V as the representation of G if it is clear what the underlying homomorphism is. Furthermore, if it is clear what the representation is from context, we will use g instead of $\phi(g)$.

Given an algebraic object, one of the first things we do is to study its subobjects. In the case of group representations, in order to build a subrepresentation, we first need to determine which subspaces are fixed under the action of the group.

Definition 2.1.2 (Invariant Subspace)**.** Let $\phi : G \to GL(V)$ be a representation. If $W \subset V$ such that $gw \in W$ for all $w \in W$, then W is invariant under the action of G.

Definition 2.1.3 (Subrepresentation)**.** Suppose $W \subset V$ is invariant under G. We call $\psi : G \to GL(W)$ where $\psi(g) = \phi(g)$ a subrepresentation of V.

Definition 2.1.4 (Irreducible Representation)**.** If the only G-invariant subspaces of V are $\{0\}$ and V, then we call ϕ an irreducible representation. Furthermore, if $V = \oplus V_i$ where the V_i are G-invariant subspaces and the subrepresentations $\phi|_{V_i}$ are irreducible, then ϕ is completely reducible.

Definition 2.1.5 (Decomposable Representation)**.** A representation V of G is decomposable if $V = V_1 \oplus V_2$ where V_1, V_2 are non-zero G-invariant subspaces. If this is not true, then V is called indecomposable.

Example 2.1.1 (Trivial Representation)**.** For a finite group G, the trivial representation is a map

$$\phi : G \to \mathbb{C}^*$$

given by $\phi(g) = 1$ for all $g \in G$.

Now for a finite group G and a set X, consider the action

$$\sigma : G \to \mathrm{Aut}(X)$$

where G acts by permuting the elements of X. A natural way to extend this action to a representation is to let X be basis elements for a vector space $\mathbb{C}X = \{\sum_{x\in X} c_x x \mid c_x \in \mathbb{C}\}$. Since a linear map is completely determined by where it sends the basis elements, G acts on $\mathbb{C}X$ by permuting the basis elements. The regular representation is a special case of this type of representation.

Example 2.1.2 (Regular Representation)**.** Let G be some finite set. Consider the vector space

$$\mathbb{C}G = \left\{ \sum_{g \in G} c_g g \mid c_g \in \mathbb{C} \right\}$$

So G is a basis for $\mathbb{C}G$. The regular representation of G is a homomorphism $R : G \to GL(\mathbb{C}G)$ where

$$R_g \left(\sum_{h \in G} c_h h \right) = \sum_{h \in G} c_h g h$$

Note that the action of G on the basis elements is the same as the action of G on itself. This representation is especially interesting, since it contains information about every irreducible representation of G.

A natural question to ask is when are two representations equivalent? One way to resolve this question is to examine linear maps between the underlying vector spaces.

Definition 2.1.6 (Morphism of Representations)**.** Given two representations, $\phi : G \to GL(V)$ and $\psi : G \to GL(W)$, a morphism from ϕ to ψ is a linear map $L : V \to W$ such that the following diagram commutes

$$\begin{tikzcd} V \arrow[r, "L"] \arrow[d, "\phi_g"] & W \arrow[d, "\psi_g"] \\ V \arrow[r, "L"] & W \end{tikzcd}$$

The set of all such linear maps is denoted as $Hom_G(\phi, \psi)$. Observe that $Hom_G(\phi, \psi) \subset Hom(V, W)$. If L is an isomorphism, then we say that ϕ and ψ are equivalent representations. This is essentially just a change of basis, since $\phi_g = L \circ \psi_g \circ L^{-1}$. Now if ϕ and ψ are equivalent representations, what types of maps will be in $\mathrm{Hom}_G(\phi, \psi)$?

Lemma 2.1.1 (Schur's lemma)**.** Let G be a finite group and let ϕ, ψ be irreducible representations of G. Suppose $T \in \mathrm{Hom}_G(\phi, \psi)$, then T is an isomorphism or the zero map. Furthermore, if $\phi \not\sim \psi$, then $\mathrm{Hom}_G(\phi, \psi) = 0$ and if $\phi = \psi$, then $T = \lambda I$ for some $\lambda \in \mathbb{C}$.

Definition 2.1.7 (Direct Sum of Representations)**.** If $\phi : G \to GL(V)$ and $\psi : G \to GL(W)$ are two representations, then their direct sum is

$$\phi \oplus \psi : G \to GL(V \oplus W)$$

where $(\phi \oplus \psi)_g(v, w) = (\phi_g(v), \psi_g(w))$.

An intuitive way to see this is to simply view the direct sum as a block diagonal matrix. For example, if $\phi_1 : G \to GL_n(\mathbb{C})$, $\phi_2 : G \to GL_m(\mathbb{C})$ are two group representations, then $\phi_1 \oplus \phi_2 : G \to GL_{m+n}(\mathbb{C})$ where

$$(\phi_1 \oplus \phi_2)(g) = \left[\begin{array}{c|c} \phi_1(g) & 0 \\ \hline 0 & \phi_2(g) \end{array}\right]$$

It is natural to ask if we can always decompose a representation into irreducible representations. Fortunately for us it is, however, to prove it we need the following notion:

Definition 2.1.8 (Unitary Representation)**.** Let V be a vector space with an inner product $\langle , \rangle$. A representation $\phi : G \to GL(V)$ is unitary if and only if

$$\langle \phi_g(v), \phi_g(w) \rangle = \langle v, w \rangle$$

for all $g \in G$ and $v, w \in V$.

The reader may have seen a similar concept in their linear algebra class. Unitary(also known as orthogonal in the case of $\mathbb{R}$) matrices preserve the inner product under transformation. Thus, a unitary representation is simply

$$\phi : G \to U_n(\mathbb{C})$$

where $U_n(\mathbb{C}) \subset GL_n(\mathbb{C})$ is the group of unitary matrices.

Proposition 2.1.1. Every unitary representation is either irreducible or decomposable.

The proof of this is trivial and left as an exercise to the reader.

It is simple to show that every representation for a finite group G is equivalent to a unitary representation, as this boils down to making an appropriate change of basis. Thus, every representation for a finite group can be decomposed into irreducible representations.

Let $\phi : G \to GL(V)$ be a representation where V is a vector space over $\mathbb{C}$. We can choose a basis for V, so that $V \simeq GL_n(\mathbb{C})$. Thus, we also have a representation $\phi : G \to GL_n(\mathbb{C})$. Observe that this gives us n^2 functions $\phi_{ij} : G \to \mathbb{C}$. What can irreducible representations tell us about the space of all functions from G to $\mathbb{C}$?

Definition 2.1.9 (Group Ring)**.** If G is a finite group, then the group ring is defined to be

$$\mathbb{C}[G] = \{f \mid f : G \to \mathbb{C}\}$$

$\mathbb{C}[G]$ is an inner product space with pointwise addition and scalar multiplication. Furthermore, the inner product is defined to be

$$\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)}$$

Multiplication is defined to be

$$(f * h)(g') = \sum_{g \in G} f(g) h(g^{-1} g')$$

2.2 Character Theory

It is often preferable to work with the character of the representation, since they contain important properties of the representation while being easier to work with. The reader may have already come across a special case of characters in the inverse discrete Fourier transform

$$x_n = \frac{1}{N}\sum_{k=0}^{n-1} X_k e^{2\pi ikn/N}$$

In this instance, the characters are the roots of unity and the function is defined over a cyclic group.

Definition 2.2.1 (Character)**.** The character of a representation $\phi : G \to GL(V)$ is defined to be map $\chi_\phi : G \to \mathbb{C}$ where $\chi_\phi(g) = \mathrm{Tr}(\phi(g))$. If the representation is irreducible, then its character is known as an irreducible character. It is important to note that a character does not necessarily have to be a group homomorphism, although in the case of abelian groups, characters are homomorphisms.

Characters are extremely useful for studying group representations, since a representation is completely determined by its character. In other words, if ϕ, ψ are group representations, then $\phi \sim \psi$ if and only if $\chi_\phi = \chi_\phi$.

Another interesting property that characters have is that they are constant on conjugacy classes, i.e. $\chi(hgh^{-1}) = \chi(g)$ for all $g, h \in G$. This property can be derived from the fact that similar matrices have the same trace. Functions that have this property are known as class functions.

The space of all class functions $Z(\mathbb{C}[G])$ is a subspace of $\mathbb{C}[G] = \{f \mid f : G \to \mathbb{C}\}$. Furthermore, the set of irreducible characters form an orthonormal basis for $Z(\mathbb{C}[G])$.

Characters satisfy interesting properties with respect to this inner product. Asides from the fact that irreducible characters are orthonormal, we have that

Theorem 1. Let C_1, C_2 be conjugacy classes for a group G and suppose $g \in C_1, h \in C_2$. Then,

$$\sum_{\chi_i} \chi_i(g)\overline{\chi_i(h)} = \begin{cases} |G|/|C_1| & C_1 = C_2 \\ 0 & C_1 \neq C_2 \end{cases}$$

3 Frobenius Reciprocity

Suppose that $\phi : G \to GL(V)$ is a group representation. If $H \leq G$, it seems natural to construct a representation on H by considering $\phi|_H : H \to GL(V)$. However, if $G \leq H$, is there a way for us to construct a representation on H from ϕ?

Definition 3.0.1 (The Induced Representation)**.** Let $\phi : H \to GL(V)$ where $H \leq G$. We first consider the vector space $\mathrm{Hom}(G, V)$ where addition and scalar multiplication are defined pointwise. Observe that there is a natural action of G on this space where $gf(g') = f(g'g)$. The induced representation is defined to be the set of maps that commute with the action of H, i.e.

$$\mathrm{Ind}_H^G(V) := \mathrm{Hom}_H(G, V)$$

As one might expect, the action of G on this space is analogous to the action of G on the set of right cosets $H\backslash G$. To see this, observe that any $f \in \mathrm{Hom}_H(G, V)$ on a coset $f(Hg)$ is completely determined by $f(g)$, since $f(hg) = \phi(h)f(g)$. Thus, if we have any $f(g) \in V$, we can extend this to a function $f : Hg \to V$ where $ha \mapsto \phi(h)(f(a))$. Thus, $\mathrm{Ind}_H^G V \simeq \mathrm{Hom}(H\backslash G, W) \simeq \oplus_{g \in H\backslash G} W_g$. Observe that the action of G permutes these subspaces by $gW_a = W_{ag^{-1}}$.

Furthermore, if χ is the character of ϕ, the induced character is given by

$$\mathrm{Ind}_H^G \chi(g) = \frac{1}{|H|} \sum_{h \in G | hgh^{-1} \in H} \chi(hgh^{-1})$$

As one might expect, the induced character is the same as the character of the induced representation. The reader may derive $\mathrm{Ind}_H^G \chi(g)$ explicitely by observing that if $Hg_ig^{-1} \neq Hg_i$, then the trace will be zero, since g will permute the W_g subspaces resulting in a block permutation matrix. Thus, we only need to consider $g_i \in H\backslash G$ such that $g_igg_i^{-1} \in H$.

Definition 3.0.2 (Restricted Representation)**.** The restricted representation is simply the function

$$\phi|_H : H \to GL(V)$$

Theorem 2 (Frobenius Reciprocity). Suppose G is a group and let H be a subgroup of G. Furthermore, let χ_H be a character for H and χ_G a character for G. Then

$$\langle \mathrm{Ind}_H^G \chi_H, \chi_G \rangle = \langle \chi_H, \mathrm{Res}_H^G \chi_G \rangle$$

holds.

Proof. By definition we have that

$$\begin{aligned}
\langle \mathrm{Ind}_H^G \chi_H, \chi_G \rangle &= \frac{1}{|G|} \sum_{g \in G} \mathrm{Ind}_H^G \chi_H(g) \overline{\chi_G(g)} \\
&= \frac{1}{|G|} \sum_{g \in G} \frac{1}{|H|} \sum_{h \in G} \chi_H(hgh^{-1}) \overline{\chi_G(g)} \\
&= \frac{1}{|G|} \sum_{g \in G} \frac{1}{|H|} \sum_{h \in G} \chi_H(hgh^{-1}) \chi_G(hg^{-1}h^{-1}) \\
&= \frac{1}{|G|} \sum_{g \in G} \frac{1}{|H|} \sum_{g \in H} \chi_H(g) \chi_G(g^{-1}) \\
&= \frac{1}{|H|} \sum_{g \in H} \chi_H(g) \overline{\chi_G|_H(g)} \\
&= \langle \chi_H, \mathrm{Res}_H^G \chi_G \rangle
\end{aligned}$$

□

Now suppose that W is an irreducible representation of a subgroup $H \leq G$. Under what conditions is $\mathrm{Ind}_H^G W$ an irreducible representation of G? Using Frobenius Reciprocity one can deduce the necessary and sufficient conditions for which $\mathrm{Ind}_H^G W$ is irreducible.

4 A View from Category Theory

4.1 A Note on Tensor Products

Suppose M, N are modules over some commutative ring R. The tensor product is an R-module $M \otimes_R N$ with an R-bilinear map

$$\otimes : M \times N \to M \otimes_R N$$

such that every R-bilinear map $\phi : M \times N \to P$ factors uniquely through $\otimes$.

$$\begin{tikzcd} M \times N \arrow[r, "\phi"] \arrow[d, "\otimes"'] & P \\ M \otimes_R N \arrow[ur, "\exists! \rho"'] & \end{tikzcd}$$

Another way to define the tensor product is $F(M \times N)/\sim$, where $F(M \times N)$ is the free $R-$module generated by $M \times N$. The tensor product is generated by elements of the form $e_m \otimes e_n$ where e_m, e_n are basis elements for M and N. The equivalence relation has the properties of:

- **Distributivity** $(m, n) + (m', n) \sim (m + m', n)$ and $(m, n) + (m, n') \sim (m, n + n')$
- **Scalar multiples** $r(m, n) \sim (rm, n) \sim (m, rn)$

The induced representation can also be constructed using tensor products. However, we first need to transform our group representations into modules! Fortunately, given a representation $\rho : G \to GL(V)$ where V is a vector space over $\mathbb{C}$, V will be a $\mathbb{C}[G]$-module. This can be seen by extending ρ linearly to

$$\rho' : \mathbb{C}[G] \to \mathrm{End}(V)$$

where $\rho'(\delta_g) = \rho(g)$. Thus, V has scalar multiplication defined by evaluation.

Let $H \leq G$ and suppose $\phi : H \to GL(V)$ is a representation. Then $\mathrm{Ind}_H^G V \cong \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$ as $\mathbb{C}[G]$-modules.

We can define a canonical isomorphism where $(f \otimes v)(g) \mapsto f(g)v$. By construction, this map will commute with the H action, since if $h \in \mathbb{C}[H]$, then $hf \otimes v = f \otimes h(v)$. In fact, this is a special case of a property called the tensor-hom adjunction.

4.2 Adjunction

Frobenius Reciprocity is a special case of a more general relationship known as adjunction. In this section, we will go over some basic category theory in order to view Frobenius Reciprocity from this more general setting.

Definition 4.2.1 (Category)**.** a category C contains the following data:

- a class ob(C) of objects
- a class hom(C) of morphisms between pairs of objects $f : X \to Y$. The set of all such morphisms between these two objects is denoted as $\mathrm{Hom}_C(X, Y)$.
- whenever the codomain of one morphism is the same as the other, there is another morphism that is defined to be their composition, i.e. if $f : X \to Y$, $g : Y \to Z$, there is another morphism $g \circ f : X \to Z$. There always exists an identity morphism $\mathrm{id}_X : X \to X$ which satisfies $\mathrm{id}_Y \circ f = f = f \circ \mathrm{id}_X$ for any $f : X \to Y$.

Furthermore, an isomorphism is a morphism $f : X \to Y$ such that there exists a unique morphism $g : Y \to X$ where $f \circ g = \mathrm{id}_X$ and $g \circ f = \mathrm{id}_Y$.

A simple example of a category would be the category **Grp** where the objects are groups and morphisms are group homomorphisms. In fact, you can define a group as a category with only one object and all morphisms are isomorphisms.

If categories contain morphisms, is there a notion of "maps" between categories?

Definition 4.2.2 ((covariant) Functor)**.** A covariant functor F from a category C to a category D contains the following data:
a map F : obj(C) $\to$ obj(D)
for every morphism $f \in \mathrm{Hom}_C(a, b)$, there exists a morphism $F(f) : F(a) \to F(b)$. We require that functors preserve the identity morphism as well as composition.

A particularly trivial example of a functor would be the functor known as the forgetful functor. This functor essentially "forgets" the structure of the morphism and object before mapping it to the output category. For example, we can define a forgetful functor F : **Grp** $\to$ **Set** which sends a group to its underlying set and group morphisms to set maps. Another example would be the identity functor id_C on a category C, which does what you would expect it to.

A question one might have is when are two categories isomorphic, i.e. for categories C, D, are there functors, $F : C \longleftrightarrow D : G$ such that $FG = \mathrm{id}_D$ and $GF = \mathrm{id}_C$? Often times the answer will be no. However, we do have the notion of equivalent categories, which is defined using natural transformations.

Definition 4.2.3 (Natural Transformation)**.** Given functors $F, G : C \to D$, a natural transformation $m : F \longrightarrow G$ consists of the following data:
for every $x \in \mathrm{ob}(C)$ there is a morphism $m(x) : F(x) \to G(x)$ such that the following

diagram commutes:

$$\begin{array}{ccc} F(x) & \xrightarrow{m(x)} & G(x) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(y) & \xrightarrow[m(y)]{} & G(y) \end{array}$$

If for all $x \in \mathrm{ob}(C)$, the morphism $m(x)$ is an isomorphism, we say that C and D are equivalent categories.

Now suppose we had functors $F : C \longleftrightarrow D : G$ from before. Instead of more strict isomorphism requirement, what if FG and GF had natural transformations with the identity on both categories? Although these categories may not be isomorphic or equivalent, these functors still have an interesting relationship known as adjunction. Equivalently, for all $y \in C, x \in D$, we have that

$$\mathrm{Hom}_D(F(x), y) \cong \mathrm{Hom}_C(x, G(y))$$

An example of the adjoint functors would be the free and forgetful functors on groups and sets. Let $G : \mathbf{Grp} \to \mathbf{Set}$ be the forgetful functor and let $F : \mathbf{Set} \to textbfGrp$ be the free functor which maps sets to the free group generated by that set. If A is a set, then any group homomorphism $F(A) \to B$ is completely determined by where it sends A as a set map! Equivalently, if we take a set map $A \to G(B)$, then we can produce a group homomorphism on $F(A) \to B$. The free-forgetful adjunction appears throughout mathematics.

Recall from the previous section that $\mathrm{Ind}_H^G V \cong \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$. Thus, induction is actually a functor $\mathrm{Rep}H \to \mathrm{Rep}G$ and restriction is its adjoint functor. Thus, we get the more general tensor-hom adjunction

$$\mathrm{Hom}(Y \otimes X, Z) \cong \mathrm{Hom}(Y, \mathrm{Hom}(X, Z))$$

4.3 Restriction and extension of scalars

As it turns out, the tensor-hom adjunction in the case of Frobenius reciprocity is simply a special case of restriction and extension of scalars. Let R, S be commutative rings and let $f : R \to S$ be a ring homomorphism. Let N be an S-module. It is nugatory to turn N into an R-module by letting $rn = f(r)n$. Observe that $N \cong \mathrm{Hom}_S(S, N)$ as R-modules. Thus, given $f : R \to S$, we have a covariant functor **S-Mod** $\to$ **R-mod**. This functor is known as the restriction of scalars, since if f is injective, we are essentially "restricting" scalar multiplication to a subring of S.

In the case of extension of scalars, we wish to create a functor from **R-mod** $\to$ **S-mod**. If M is an R-module, we can create an S-module $M \otimes_R S$. It is easy to verify using the properties of tensor products that this is an S-module. Furthermore, we note that extension and restriction of scalars are adjoint functors, i.e.

$$\mathrm{Hom}_R(M, \mathrm{Hom}_S(S, N)) \to \mathrm{Hom}_S(M \otimes_R S, N)$$

Observe that $\iota : \mathbb{C}[H] \to \mathbb{C}[G]$ is a ring homomorphism via inclusion. Thus, if V is a representation of G, we can turn it into a representation of H where $hv = \iota(h)v$. Now suppose W is a representation of H. Observe that we get that

$$\mathrm{Hom}_{\mathbb{C}[H]}(W, \mathrm{Res}_H^G V) \cong \mathrm{Hom}_{\mathbb{C}[H]}(W, V) \cong \mathrm{Hom}_{\mathbb{C}[G]}(W \otimes_{\mathbb{C}[H]} \mathbb{C}[G], V) \cong \mathrm{Hom}_{\mathbb{C}[G]}(\mathrm{Ind}_H^G W, V)$$

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